AN EXCHANGE MODEL WITH UNCERTAIN COMMODITIES

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Introduction

The case of an uncertain future is a realistic assumption to make in studying the optimum allocation of resources. Although many interesting results, related to this subject, have come into light in recent years, we are still far from having a well established theory of uncertainty.

An interesting approach to the problem is the one of K. Borch, exposed in his papers "The Economics of Uncertainty",* "Reformulation of Some Problems in the Theory of Risk"** and "Payment of Dividend by Insurance Companies".*** As an introduction to the present paper we are going to present in brief some of his views contained in "The Economics of Uncertainty". For the time being we are interested in the simple exchange model which he uses. Later on we will discuss several of his ideas related to the extension of the present paper.

There are two individuals, A and B. Each of them is assumed to have a utility function of money which is quadratic. Utility is quasi-measurable, i.e., only differences in utility levels are meaningful. Individual A possesses a business which will yield profit x_1 with probability p_1, or profit zero with probability $(1-p_1)$. Denote the returns of business 1 by $(x_1p_1, 0(1-p_1))$. Similarly B possesses a business which will yield $(x_2p_2, 0(1-p_2))$. According to Bernelli's principle, individual j (=1,2) assigns his business the utility

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*Econometric Research Program, Research Memorandum, No. 54, 20 March 1963.
**Econometric Research Program, Research Memorandum, No. 46, 8 October, 1962.
***Econometric Research Program, Research Memorandum, No. 51, 15 January 1962.

$$U_j = (1-p_j)\, u_j(0) + p_j u_j(x_j) \qquad j = 1,2$$

where $u_j(x)$ is the utility of money function of the jth individual. It is easy to see that A and B can increase their utility by exchanging shares of the businesses they possess.

Let $\alpha_j = (\alpha_{1j}, \alpha_{2j})$ indicate the proportion of shares of businesses 1 and 2 that individual j holds after an exchange has taken place. The utility which individual j assigns to his portfolio of shares after the exchange is given by

$$U_j(\alpha_{1j}, \alpha_{2j}) = (1-p_1)(1-p_2)u_j(0) + p_1(1-p_2)u_j(\alpha_{1j}x_1) +$$
$$+ p_2(1-p_1)u_j(\alpha_{2j}x_2) + p_1p_2u_j(\alpha_{1j}x_1+\alpha_{2j}x_2) \qquad j = 1,2$$

where each term indicates the expected utility of the profits of individual j in each of the possible "states of the world".

Under the assumption of rationality a portfolio composition $(\alpha_{11}, \alpha_{21})$ for A and $(1-\alpha_{11}, 1-\alpha_{21})$ for B is acceptable by both individuals if it satisfies the following conditions.

a) $U_1(\alpha_{11},\alpha_{21}) > U_1(1,0)$ (1)

$U_2(1-\alpha_{11}, 1-\alpha_{21}) > U_2(0,1)$

i.e. after an exchange both individuals have higher utility than before it, and

b) There exists no $(\alpha^o_{11}, \alpha^o_{21})$ such that

$$U_1(\alpha^o_{11}, \alpha^o_{21}) > U_1(\alpha_{11}, \alpha_{21})$$
$$U_2(1-\alpha^o_{11}, 1-\alpha^o_{21}) > U_2(1-\alpha_{11}, 1-\alpha_{21})$$

i.e. there is no other composition of portfolios which gives both individuals higher utility.

Borch derives a relation of the components of all vectors $(\alpha_{11}, \alpha_{21})$ which satisfy condition (b). This relation, which in the case of two kinds of shares guarantees that the Jacobian $\frac{\partial(U_1, U_2)}{\partial(\alpha_{1j}, \alpha_{2j})}$ $j = 1,2$ is zero, together with condition (a) determine a set of optimal exchange arrangements, or equivalently, of optimal portfolio selections.

Later on we will come back to the paper of Borch. For the time being let us consider this exchange model in greater detail.

Chapter 1.

Our purpose is to show that if the individuals A and B maximize their expected utility corresponding to an exchange ratio Π of shares 1 per share 2, we can specify a vector $(\alpha^*_{11}, \alpha^*_{21})$, out of the set of optimal exchange arrangements, on the basis of which the exchange will be realized. In addition to this the equilibrium $\alpha^*_{21} > 0$ and $\alpha^*_{12} > 0$. This means that the case of exchange is always preferred to the case of non-exchange. We can also determine an equilibrium price Π^* of shares 1 per share 2, depending on the kinds of shares existing and the attitude of individuals towards risk.

Let $u_1(m) = \mu m - \lambda m^2$ and $u_2(m) = Km - \Theta m^2$ be the utility functions of A and B respectively. Assume that the preference orderings described by $u_j(m)$ satisfy the von Neumann-Morgenstern axioms of choice among risky prospects.

Assumptions a) μ, λ, K, Θ all greater than zero

b) $\mu/2\lambda \geq x_1 + x_2$ and $K/2\theta \geq x_1 + x_2$

c) $0 < p_1, p_2 < 1$.

Assumption (a) guarantees that both utility functions are concave with respect to increasing amounts of money; (b) is the assumption of non-saturation and by (c) we require the presence of uncertainty. Assumption (b) also establishes the equivalence between a probability distribution in terms of the variance and the expected mean, and the same distribution in terms of p_i and x_i.(1)

Let $s = (p_1x_1, p_2x_2)$ be the "initial state of the world". By this term we mean the vector indicating the profits of each business and the corresponding probabilities, before any exchange takes place. Let α_j, where j refers to an individual, be as before. Let F* be the set of two-parameter probability distributions, and Π an exchange ratio of shares 1 per share of 2. Consider individual j. For given s and any vector α_j^i (where j refers to the individual and i refers to any particular vector, i = 1, 2 ...), the pair $(\alpha_j^i; s)$ maps into a subset F_j of F*. Let $f_j^i(\alpha_j^i) \in F_j$, j = 1,2, i = 1,2, ... Here f_j^i is the two-parameter probability distribution associated with a_j^i. Because of the assumption about the nature of the orderings described by $u_j(m)$ and the von Neumann-Morgenstern theorem we know that the functional

$$U[f(\alpha_j^i, m)] = \int_{-\infty}^{+\infty} u_j(m) f(\alpha_j^i, m)\, dm \qquad j = 1,2, \qquad i = 1,2 \ldots$$

where m refers to profit,

(1) K. Borch: "A Note on Utility and Attitudes towards Risk". There is shown that some of the assumptions related to the probability distributions in terms of expectation and variance, and in terms of p_i and x_i contradict each other when assumption (b) does not hold.

describes the preference ordering of the jth individual over the set of feasible probability distributions $f_j^i(\alpha_j^i)$.

Thus given an exchange ratio Π we can choose a probability distribution which is preferred to all other probability distributions satisfying the budget constraint imposed by Π. To this probability distribution corresponds an $\tilde{\alpha}_j$ [2] which is optimum for the given Π.

In other words for given Π, individual A acts as follows:

$$\max \int u_1(\alpha_1^i) f_1^i(\alpha_1^i) d\alpha_1^i \qquad i = 1,2, \ldots \qquad (2)$$

$$\text{subject to } \Pi\alpha_{11} + \alpha_{21}^* = \Pi.$$

In the same way individual B

$$\max \int u_2(\alpha_2^i) f_2^i(\alpha_2^i) d\alpha_2^i, \qquad (3)$$

$$\text{subject to } \Pi\alpha_{12} + \alpha_{22} = 1.$$

Thus for each individual we obtain a vector $\tilde{\alpha}_j (j = 1,2)$ indicating his optimum portfolio composition corresponding to a given exchange ratio. It is very easy to show that if an $\tilde{\alpha}_j$ is a solution to the above maximization problem, the corresponding probability distribution $f_j(\alpha_j) \in \tilde{F}_j$, where $\tilde{F}_j = \left\{ f_j^i(\alpha_j^i) \mid U_j(f_j^i) > U_j(f_j^o) \right\}$ $j = 1,2$, and where $U_j(f_j^o)$ is

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(2) For the case of two commodities we suspect, but have not proved, that the choice of $\tilde{\alpha}_j$ is unique. It also seems that the inverse mapping (call if $\phi^{-1} : \alpha_j \to f_j \in F_j$) exists and it is continuous. This means that to a given probability distribution corresponds a unique choice of the portfolio and the other way round. The proof of these statements, if they are true, will appear in the extension of the present paper. For more than two commodities these, of course, cannot be true.

the utility assigned by individual j to the probability distribution corresponding to his holdings of share j before any exchange. Therefore any such $\tilde{\alpha}_j$ satisfies condition (a) in (1). Also condition (b) is satisfied by the definition of $\tilde{\alpha}_j$ as a solution to a maximization problem. Thus the set of $\tilde{\alpha}_j$'s (the set of optimum portfolio compositions corresponding to all possible exchange ratios) satisfies both conditions of Borch's optimal solutions. But not all of them, or maybe none, can be equilibrium points since they may not be feasible. At an equilibrium point we must have in addition:

$$\tilde{\alpha}_{21} = 1-\tilde{\alpha}_{22} \qquad (4)$$

$1-\tilde{\alpha}_{11} = \tilde{\alpha}_{12}$ i.e. demand equals supply for both shares.(3)

Let E be the set of pairs of vectors $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ which are solutions to the above maximization problems and satisfy (4).

Proposition 1. Under assumptions (a)-(c), $E \neq \phi$.

Instead of a proof we will establish the conditions which make applicable the proof of existence given by D. Gale and H. Nikaido.* We will do this for the general case in which there are n individuals, each one possessing a business which will yield $(x_j p_j, O(1-p_j))$, $j = 1 \ldots n$. What is being traded in this model is proportions of shares of each business. We will concentrate on the equilibrium of Demand and Supply of shares, which guarantees the equilibrium of Demand and Supply

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(3) One of the two relations in (4) holds automatically by Walras' law as long as the other holds.

* D. Gale, "The Law of Supply and Demand", Math. Scand. 1955 (155-167). H. Nikaido, "On the Classical Multilateral Exchange Problem", Metroeconomica, 1955 (135-145). Both authors prove the existence of a competitive equilibrium along the same lines.

of expected returns.

Let A_j, $j = 1, \ldots, n$, be the commodity space of the jth individual. More explicitly A_j is the space whose ith, $i = 1 \ldots n$, coordinate measures proportions of the ith share. A_j is assumed to be convex and compact, and contain the null vector, as Assumption I of Gale's paper requires.

Let $P = \left\{\Pi \mid \Pi_i \leq 0, \Sigma\Pi_i = 1\right\}$ be the n-1 price simplex. Let $\alpha_j = (-\alpha_{ij}, \ldots, 1-\alpha_{jj}, \ldots, -\alpha_{nj}) \in A_j$ be the supply vector of the jth individual. The negative sign indicates that the relevant share is demanded rather than supplied. Given a price vector $\Pi \in P$, the individual chooses a vector $\alpha_j \in A_j$ which "maximizes his staisfaction". We quote Assumption II: Corresponding to the jth individual "there is a function S_j, called the jth unit's supply function, defined for each price vector Π in P and such that $S_j(\Pi)$ is a subset of A_j subject to the following restrictions.

(1) $S_j(\Pi)$ is a nonempty convex subset of A_j.

(2) If $\alpha_j \in S_j(\Pi)$ then $\alpha_j \Pi \geqq 0$.

(3) S_j is continuous in the sense that the graph of S_j is a closed subset of $P \times A_j$

We will show that restrictions (1)-(3) hold for our case. In doing so we will use the theorems and definitions used by D. Gale in section 7 of the above mentioned paper.

i) The preference ordering described by the utility

function $U_j(\alpha_{ij}, \ldots \alpha_{nj})$ is a continuous ordering. For the jth individual $U_j(\alpha_{ij} \ldots \alpha_{nj}) = \sum_{k=1}^{2^n} \bar{P}_k \, u_j(\alpha_j^{(k)})$

$$= \mu_j\left(\sum_{i=1}^{n} p_i x_i \alpha_{ij}\right) - \lambda_j\left(\sum_{i=1}^{n} p_i(1-p_i)x_i^2 \alpha_{ij}^2\right) - \lambda_j\left(\sum_{i=1}^{n} p_i x_i \alpha_{ij}\right) \quad j=1 \ldots n,$$

where $\bar{p}_k$ is the conditional probability of the kth "state of the world", $\alpha_j^{(k)}$ is the part of α_j corresponding to the kth state, and μ_j and λ_j are defined in assumptions (a) and (b) above.

U_j is continuous with respect to $\alpha_j = (\alpha_{ij} \ldots \alpha_{nj})$ since it is the sum of a polynomial and the product of two polynomials in α_{ij}, (i = 1 ... n, for a fixed j).

(ii) To show that the above continuous ordering is also convex. Notice that U_j is a concave function since it consists of the sum of concave functions.

Let $S^+(\alpha) = \{\alpha' | \alpha'(\geq)\alpha\}$ for α, $\alpha' \in A$. (Here we have omitted the subscripts.) By $(\geq)$ we mean, a) for any two α, $\beta \in A$, either $\alpha(\geq)\beta$, or $b(\geq)\alpha$. If both hold we write $\alpha(\sim)\beta$. If $\alpha(\geq)\beta$ and $\alpha(\not\sim)\beta$, we write $\alpha\,(>)\beta$. b) The relation is transitive. To show that the ordering is continuous it is enough to show that $S^+(\alpha)$ is convex for all $\alpha \in A$. (Lemma 5).

Let $\beta, \gamma \in S^+(\alpha)$. Assume $S^+(\alpha)$ not convex. Then there is a δ in the segment $[\beta,\gamma]$, such that $\delta(<)\alpha$, i.e., $U(\delta) < U(\alpha)$. By concavity of U we have:

$U(\delta) = U(t\beta + (1-t)\gamma) \geq t\,U(\beta) + (1-t)U(\gamma)$ for some $0 \leq t \leq 1$.

But $\beta, \gamma \in S^+(\alpha)$ implies $U(\beta) \geq U(\alpha)$ and $U(\gamma) \geq U(\alpha)$. Substituting above we have $U(\delta) = U(t\beta + (1-t)\gamma) \geq tU(\beta) + (1+t)U(\gamma)$ $= U(\alpha)$. This is a contradiction to the conclusion we reached

above. $U(\delta) < U(\alpha)$. Therefore $S^{+}(\alpha)$ is convex and the ordering $U_j(\alpha_{ij} \ldots \alpha_{nj})$ is convex. Then (by Theorem 5,) $S_j(\Pi)$ is convex for all $\Pi \in P$.

Condition (2) of Assumption II, is satisfied as one can see from (2) and (3) above.

Condition (3) of Assumption II requires that each individual can supply a positive amount of each share. This requirement does not hold in our model. Instead we follow the closed extension procedure used by Nikaido in the supplementary note to "On the Classical Multilateral Exchange Problem"**. The assumption behind is that each individual possesses some commodity while all the individuals taken together can supply every commodity, which certainly holds in our case. Nikaido, under the above assumptions and the non-saturation assumption, established in our model by assumption (b), proves the existence of an equilibrium price vector Π^* which is positive.

Analytically we can obtain an expression of the equilibrium exchange ratio Π^* and the equilibrium vectors α_1^* and α_2^*. Solving the maximization problem (2) by substituting in $U_1(\alpha_{11}\alpha_{21})$ for $\alpha_{21} = (1-\alpha_{11})\Pi$ and then differentiating with respect to α_{11}, we obtain

$$\tilde{\alpha}_{11} = \frac{\frac{\mu}{2\lambda}(p_1x_1 - p_2x_2\Pi) + p_2x_2\Pi(x_2\Pi - p_1x_1)}{p_1x_1(x_1 - p_2x_2\Pi) + p_2x_2\Pi(x_2\Pi - p_1x_1)} .$$

** Metroeconomica, 1957, p.209-210.

Also from (3) we obtain

$$\tilde{\alpha}_{12} = \frac{\frac{K}{2\theta}(p_1x_1 - p_2x_2^{\Pi}) + p_2x_2(x_2^{\Pi} - p_1x_1)}{p_1x_1(x_1 - p_2x_2^{\Pi}) + p_2x_2^{\Pi}(x_2^{\Pi} - p_1x_1)} .$$

From $\tilde{\alpha}_{11}$, $\tilde{\alpha}_{12}$ and the equilibrium condition (4), namely,

$$1 - \tilde{\alpha}_{11} = \tilde{\alpha}_{12}, \quad \text{or} \quad \tilde{\alpha}_{11} + \tilde{\alpha}_{12} = 1,$$

we obtain the equilibrium price ratio Π^*.

$$\Pi^* = \frac{p_1x_1}{p_2x_2}\,\delta. \tag{5}$$

In a similar way we obtain $\tilde{\alpha}_{21}$ and $\tilde{\alpha}_{22}$. Substituting Π^* in $\tilde{\alpha}_{ij}$ we obtain the proportion of each share that an individual holds in equilibrium.

$$\alpha^*_{11} = \frac{\frac{\sigma}{p_1x_1}(1-\delta) + \delta\left(\frac{\delta}{p_2} - 1\right)}{\left(\frac{1}{p_1} - \delta\right) + \left(\frac{\delta}{p_2} - 1\right)} \tag{6}$$

$$\alpha^*_{21} = \frac{p_1x_1\left\{\left(\frac{1}{p_1} - \delta\right) - \frac{\sigma}{p_1x_1}(1-\delta)\right\}}{p_2x_2\left\{\frac{1}{\delta}\left(\frac{1}{p_1} - \delta\right) + \left(\frac{\delta}{p_2} - 1\right)\right\}} \tag{7}$$

$$\alpha^*_{12} = \frac{\frac{\phi}{p_1x_1}(1-\delta) - \frac{p_2x_2}{p_1x_1}\left(\frac{\delta}{p_2} - 1\right)}{\left(\frac{1}{p_1} - \delta\right) + \delta\left(\frac{\delta}{p_2} - 1\right)} \tag{8}$$

$$\alpha^*_{22} = \frac{\frac{1}{\delta}\left(\frac{1}{p_1} - \delta\right) - \frac{\phi}{p_2x_2}(1-\delta)}{\frac{1}{\delta}\left(\frac{1}{p_1} - \delta\right) + \left(\frac{\delta}{p_2} - 1\right)} \tag{9}$$

$$\text{where } \delta = \frac{\tau - p_2 x_2 - x_1}{\tau - p_1 x_1 - x_2} \; ; \; \sigma = \frac{\mu}{2\lambda} \; ; \; \phi = \frac{k}{2\theta} \; ;$$

$$\tau = \sigma + \phi \qquad (10)$$

From the above expressions we observe that Π^* and α^*_{ij} are well defined and in addition $\Pi^* > 0$.

To see this consider (6) and (10). δ is never zero by the nonsaturation assumption. Also the denominator of α_{ij} is:

$$(\frac{1}{p_1} - \delta) + \delta(\frac{\delta}{p_2} - 1) = p_1\delta^2 - 2p_1p_2\delta + p_2.$$

Assume this expression to be equal to zero and solve for δ.

$$\delta = \frac{2p_1p_2 \pm 2(p_1^2p_2^2 - p_1p_2)^{\frac{1}{2}}}{2p_1}$$

The expression $p_1^2p_2^2 - p_1p_2$ is less than zero since p_1, p_2 belong to the open interval (0,1) by assumption (c). Therefore the polynomial has only complex roots.

Proposition 2. Under assumptions (a)-(c), $\alpha^*_{21} > 0$ and $\alpha^*_{12} > 0$ This proposition asserts that in equilibrium there is always exchange among the individuals, or that under no conditions the individuals are satisfied with their initial holdings of shares.

Proof. We will prove proposition 2 only for α^*_{21}. A similar proof holds for α^*_{12}.

Suppose $\alpha^*_{21} \leq 0$. This implies that the numerator in (7) is equal or less than zero, or

$$\left(\frac{1}{p_1}-\delta\right) - \frac{\sigma}{p_1 x_1}(1-\delta) \leqq 0. \qquad (11)$$

This can be written as

$$\frac{1}{p_1} - \frac{\sigma}{p_1 x_1} \leqq \delta - \frac{\sigma}{p_1 x_1}\delta = \delta\left(1-\frac{\sigma}{p_1 x_1}\right)$$

By assumption (b) $1-\frac{\sigma}{p_1 x_1} < 0$. Therefore

$$\delta \leqq \frac{\sigma - x_1}{\sigma - p_1 x_1} \qquad (12)$$

From (12) and (10) we have

$$\delta = \frac{\tau - p_2 x_2 - x_1}{\tau - p_1 x_1 - x_2} = \frac{(\sigma - x_1)+(\phi - p_2 x_2)}{(\sigma - p_1 x_1)+(\phi - x_2)} \leqq \frac{\sigma - x_1}{\sigma - p_1 x_1}.$$

Using again assumption (b) we have:

$$\frac{\phi - p_2 x_2}{\phi - x_2} \leqq \frac{\sigma - x_1}{\sigma - p_1 x_1} \qquad (13)$$

Since $p_1, p_2 < 1$ by assumption (c) the right hand of (13) is less than one while the left hand term is greater than one. I.e., we have

$1 < \frac{\phi - p_2 x_2}{\phi - x_2} \leqq \frac{\sigma - x_1}{\sigma - p_1 x_1} < 1$. We have reached a contradiction by assuming that (11) is equal to zero. Therefore $\alpha^*_{21} > 0$.

From the budget constrain in (2) we have $\alpha^*_{21} = (1-\alpha^*_{11})\Pi^*$. Now $\alpha^*_{21} > 0$ implies $\alpha^*_{11} < 1$. Similarly we

obtain that $\alpha^*_{22} < 1$. Consider α^*_{11} in (6). Assume that $\alpha^*_{11} \leq 0$, i.e., $\frac{\sigma}{p_1 x_1}(1-\delta) + \delta(\frac{\delta}{p_2}-1) \leq 0$ or $\frac{p_2\sigma}{p_1 x_1}(1-\frac{1}{\delta}) \geq \delta - p_2$.

i) If $\delta < 1$. $\frac{\delta(\delta-p_2)}{\delta-1} = \frac{\delta(p_2-\delta)}{1-\delta} \geq \frac{p_2\sigma}{p_1 x_1}$

It is obvious that if $\delta > p_2$, $\frac{p_2\sigma}{p_1 x_1} \leq 0$ which is a contradiction.

Assume $\delta < p_2$. Then substitute above for p_2,

$\frac{\delta(p_2-\delta)}{1-\delta} = \frac{\delta(\delta-\delta)}{1-\delta} = 0$, and $\frac{p_2\sigma}{p_1 x_1} \leq 0$ which is again a contradiction.

ii) If $\delta = 1$, $\alpha^*_{11} \leq 0$ leads also to a contradiction because $\frac{\sigma}{p_1 x_1}(1-\delta)+\delta(\frac{\delta}{p_2}-1) \leq 0$ becomes $(\frac{1}{p_2}-1) \leq 0$. However $\frac{1}{p_2} > 1$ since $p_2 < 1$ and $(\frac{1}{p_2}-1) > 0$, always.

iii) If $\delta > 1$. For $\alpha^*_{11} \leq 0$, δ must lie

$$1 < \delta < \frac{p_2\sigma}{p_1 x_1}(1-\frac{1}{\delta}) + p_2$$

Thus a necessary and sufficient condition for $\alpha^*_{11} \leq 0$ is that

$$\frac{p_2\sigma}{p_1 x_1}(1-\frac{1}{\delta}) + p_2 > 1$$

or $$\frac{\sigma}{p_1 x_1}(1-\frac{1}{\delta}) > \frac{1-p_2}{p_2}.$$

This condition, roughly speaking, requires that, for $\alpha^*_{11} \leq 0$, we must have striking differences between the two kinds of

shares with respect to profits and probabilities. When $p_2x_2 + x_1 = p_1x_1 + x_2$ [4] in (12), $\delta=1$ and (6) becomes

$$\alpha^*_{11} = \frac{\frac{1}{p_2}-1}{(\frac{1}{p_1}-1)+(\frac{1}{p_2}-1)} \quad . \tag{14}$$

As it is easy to see from (14) in this case $0 < \alpha^*_{11} < 1$. A similar analysis holds for α^*_{22}.

The above suggests that individuals always reach their equilibrium by trading. In other words the optimum distribution of risk requires that they should diversify the composition of their portfolios. When α^*_{11} (or α^*_{21}) is negative, we can think of individual B (or A) as making a loan to A (or B) so that both are able to reach their preferred probability distribution of profits. This loan can be thought of either as a creation of new shares by B (or A), or as a promise of A (or B) to pay to B (or A) an amount $\alpha_{11}x_1$ (or $\alpha_{22}x_2$) if business 1 succeeds (or if business 2 succeeds). For the time being we won't discuss any of the problems raised by such an interpretation.

From (5) and (10) we have

$$\lim_{\tau \to \infty} \Pi^* = \frac{p_1x_1}{p_2x_2} \quad .$$

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(4) See footnote (2). It is this case that causes difficulties in proving the statements of footnote (2).

Substituting in (6) and (9) we have

$$\lim_{\tau\to\infty} \alpha^*_{11} = \frac{(\frac{1}{p_2}-1)}{(\frac{1}{p_1}-1)+(\frac{1}{p_2}-1)} \qquad \text{and}$$

$$\lim_{\tau\to\infty} \alpha_{22}\frac{(\frac{1}{p_1}-1)}{(\frac{1}{p_1}-1)+(\frac{1}{p_2}-1)}$$

Again it is easy to see that the limit of α^*_{11} and α^*_{22} lie in the open interval (0,1). This is also true for α^*_{21} and a^*_{12}. The condition that $\tau\to\infty$ corresponds to the case in which the utility function of at least one individual is approximately linear, as one can see from (12). But in the limit, equilibrium does not exist in general, as it is known. Only in the special case where $p_2x_2 + x_1 = p_1x_1+x_2$ (15) there is equilibrium. This is so because $\tau\to\infty$ implies $\delta\to 1$. In the limit, (6)-(9) become

$$(6') \quad \alpha^*_{11} = \frac{\frac{1}{p_2}-1}{(\frac{1}{p_1}-1)+(\frac{1}{p_2}-1)}$$

$$(7') \quad \alpha^*_{21} = \frac{p_1x_1(\frac{1}{p_1}-1)}{p_2x_2\left\{(\frac{1}{p_1}-1)+(\frac{1}{p_2}-1)\right\}}$$

$$(8') \quad \alpha^*_{12} = \frac{\frac{p_2x_2}{p_1x_1}(\frac{1}{p_2}-1)}{(\frac{1}{p_1}-1)+(\frac{1}{p_2}-1)}$$

$$(9') \quad \alpha^*_{22} = \frac{\frac{1}{p_1}-1}{(\frac{1}{p_2}-1)+(\frac{1}{p_1}-1)} \quad .$$

As one can see from (6')-(9'), equilibrium in general does not exist unless the x_i and p_i stand in special relation to each other. This is so because some of the α^*_{ij}'s depend on expected returns and some are independent of them.

It is interesting to notice that even when (15) holds there is no equilibrium if we allow p_1 (or p_2), but not both[5], to be equal to one. In this case also equilibrium does not exist in general unless the relation (15) holds among expected returns. But relation (15) requires that, in addition to $p_1 = 1$, (or $p_2=1$), p_2 is also equal to one (or $p_1 = 1$) and thus the model degenerates to a one commodity model, just what we were trying to avoid.

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(5) When $p_1 = p_2 = 1$ the model degenerates to a one commoduty model.

Chapter 2.

In this chapter we are going to discuss in brief some further questions related to uncertainty and indicate the extensions of this paper to be undertaken in my Ph.D. thesis.

Consider the problem of competitive allocation of resources under uncertainty. Arrow, in his "Le Role des Valeurs Boursières pour la Répartition la Meilleure des Risques", assumes a hypothetical market in which "possible rights on goods" are treated. The ith individual purchases from this market an amount of "possible right", which will provide him with the quantity x_{isc} of commodity c if state s prevails, at prices $\bar{p}_{sc}$ subject to the constraint of his income y_i. He makes his choice on the basis of a utility function which describes his preference ordering of the "possible rights" space. Then he shows that under certain conditions a competitive equilibrium in such markets leads to an optimum distribution of risks. Further he shows that this model gives the same results with a competitive model in which the individuals act as follows: Each individual spends his income y_i on speculative stocks (lottery tickets would be more appropriate). The sth stock, assuming that there are at least as many stocks as "states of the world", is bought at price g_s and will yield \$1 if the sth state prevails, or zero dollars in any other case. After a state occurs the individual uses his new income y^*_{is} to buy commodities at prices p_{sc} which are fixed only after that particular state has occurred.

Similarly K. Borch, in his above mentioned paper, "Economics of Uncertainty", shows that for each optimal vector α_j, which can be obtained through exchange of shares, we can find a solution which gives both individuals higher utility, if instead of exchanging shares the individuals agree on how they are going to split the profits in each of the "possible states of the world". This suggests that a competitive equilibrium leads to a sub-optimum, unless the number of the shares tends to infinity.(6)

It seems that the essence of both arguments given above is the ability of the individual to participate in each "state of the world" in a proportion which matches his attitide towards risk. In our simple model the introduction of "money" seems to give a higher degree of freedom in choosing the state in which one wishes to participate more porportionately.(7) More specifically, if we assume that each individual, in addition to shares, is endowed with an amount of money and that shares can be treated by means of this money, we give the seller of shares higher flexibility in favoring a particular state by deciding whether and to what extent he wishes to hold shares of the other kind. The "states of the world" are the

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(6) A justification of this can be seen as an extension of proposition 2, above.

(7) When buying shares the individual cannot decide about the proportion he wishes to participate in a particular state independently of the proportion he participates in other states. For example, he participates in the ith state, in which the ith business succeeds, with the same amount of shares of this business as in the state i+k, in which both businesses, i and k, succeed. It is in this sense that the word proportionately is used.

same as before "money" enters the picture, only now there is a higher degree of freedom in the individual's action. But again this flexibility is less than a profit-split agreement for example. In reality, though, the existence of insurance companies seems to provide the individual, theoretically at least, with extra flexibility and makes him move towards an optimum. The argument for this statement is as follows: It can be shown that a profit-split agreement is equivalent to an exchange of shares in the case in which the sum of the profits of the second and third states (where business 1 succeeds and business 2 fails, and business 1 fails and 2 succeeds, respectively), for each individual, equals the profits of the fourth state (where both businesses succeed). Call the profits of individual A in the second, third and fourth state x_2, x_3, x_4 respectively. If $x_2+x_3 = x_4$ this profit-split agreement is equivalent to an exchange of shares, as said above. If his optimum is obtained at an agreement in which the profits of each state are x_2, x_3, x_4, but $x_2+x_3 \neq x_4$, the individual can exchange shares to an amount equivalent to a profit-split agreement x_2', x_3', x_4', such that $x_2' + x_3' = x_4$, and enter an insurance contract which will provide him with $x_i - x_i' \geq 0$, $i = 2,3,4$, if the ith state occurs. Thus he corrects the disadvantage of his portfolio of shares which allows him only proportionate participation in the 2nd, 3d, and 4th state. He can probably do better than that by insuring himself for the first state (where both businesses fail). What is said above is possible, given the symmetric position of buying shares and insuring oneself, when gambling and insurance are considered

as fair games. But, there is the question whether the risk under consideration is insurable. It seems that the expansion of insurance companies into new fields of insurable risks helps the economy to move towards an optimum allocation of resources by means of free competition. If the above argument is correct the individual is expected to gamble and buy insurance at the same time without having to assume that his utility function is a polynomial of third degree. A rigorous support of the above arguments will be undertaken in my Ph.D. thesis.

Before extending this model to a dynamic one, a generalization of the results obtained so far is required. Another interesting question in the static case is what happens when the subjective probabilities p_i differ among individuals. This subject is of higher importance in the dynamic case if we allow the probabilities to change from period to period under different assumptions about "learning" and formation of expectations. The introduction of production and consumption in the model seems a natural process. There we hope to say something about dynamic and forward prices, structure of the rates of interest, and the composition of assets.

The problems involved in the proposed extensions of this paper are difficult and, some of them, formidable. We are aware of the difficulties but we hope to be able to find a way through some of them.